2020 B
April 20

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There are 2 operations on a v.f. $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$.

the curl of $\vec{F}$, curl $\vec{F}$ or $\nabla \times \vec{F}$ in notation,

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & P \end{vmatrix} \quad \text{(formally)}$$

$$= (P_y - N_z)\hat{i} - (P_x - M_z)\hat{j} + (N_x - M_y)\hat{k}$$

e.g. Let $\vec{F} = xy\hat{i} + z^2\hat{j} + 2xz\hat{k}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xy & z^2 & 2xz \end{vmatrix}$$

$$= (0 - 2z)\hat{i} - (2z - 0)\hat{j} + (0 - x)\hat{k}$$

$$= -2z\hat{i} - 2z\hat{j} - x\hat{k} \quad \#$$

the divergence of $\vec{F}$, div $\vec{F}$ or $\nabla \cdot \vec{F}$ in notation,

$$\nabla \cdot \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

e.g. $\vec{F}$ as above.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y} z^2 + \frac{\partial}{\partial z} 2xz$$

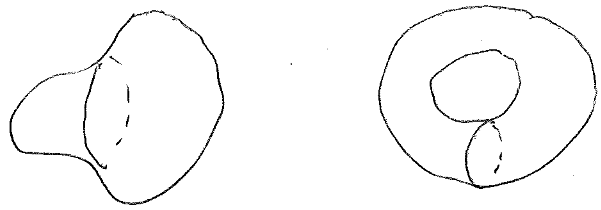
$$= y + 2x \quad \#$$

Note

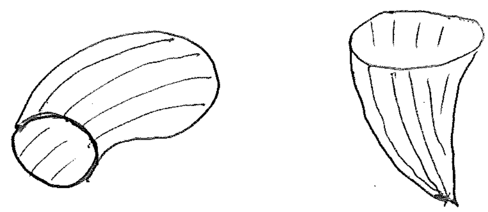
$$\text{v.f.} \xmapsto{\text{curl}} \text{v.f.}$$

$$\text{v.f.} \xmapsto{\text{div}} \text{fcn} \, .$$

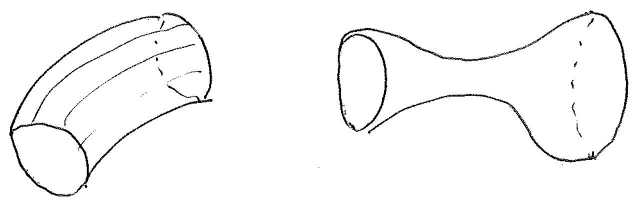
Surfaces in space can be classified by its boundary.



closed surfaces
(surfaces without boundary)



surfaces bdd by one curve



surfaces bdd by 2 curves,

<u>Stokes thm</u> Let S be a surface in space bounded by a single curve C. The orientation of C is in anticlockwise direction w.r.t. the chosen unit normal $\hat{n}$ of S. Let $\vec{F}$ be a smooth v.f. on S. Then

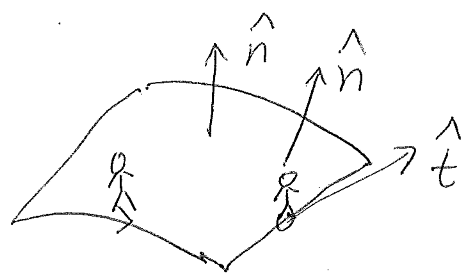
$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\, d\sigma = \oint_C \vec{F}\cdot d\vec{r}$$

Note: $\oint_C \vec{F}\cdot d\vec{r}$ is the circulation of $\vec{F}$ around C.

$\iint_S \nabla\times\vec{F}\cdot\hat{n}\, d\sigma$ is the flux of $\nabla\times\vec{F}$ across S. When a parametrization of S: $(u,v)\in D \longrightarrow \vec{r}(u,v)$ is given,

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma = \iint_D \nabla\times\vec{F}(\vec{r}(u,v))\cdot\vec{r}_u\times\vec{r}_v\,dA(u,v).$$

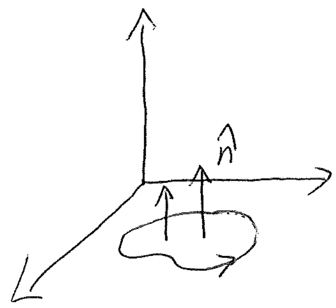
How to determine the orientation of C when the orientation of S, ie, $\hat{n}$, has been chosen?



Imagine a person walking along $\hat{t}$ whose head pointing in $\hat{n}$-direction, the surface S should lie on his/her left hand side.

Note. Stokes' thm reduces to Green's thm when S is flat, ie, let $S = \{(x,y,0) : (x,y)\in D\}$. and when
$$\vec{F} = M(x,y)\,\hat{i} + N(x,y)\,\hat{j} + 0\,\hat{k} \quad (\vec{F} \text{ is flat too}).$$

Now, $$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ M & N & 0 \end{vmatrix} = 0\,\hat{i} - 0\,\hat{j} + (N_x - M_y)\,\hat{k}$$



the surface S parametrized by
$$(x,y) \longmapsto (x,y,0),$$
ie the graph of $f(x,y)\equiv 0$.

If we take $\hat{n} = (0,0,1) = \hat{k}$, then the boundary of S, C, is anticlockwise as viewed from above (or as viewed in the xy-plane).

$$\iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma = \iint_D (N_x - M_y)\,\hat{k}\cdot\hat{k}\,dA(x,y) = \iint_D (N_x - M_y)\,dA(x,y)$$

Stokes' thm

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy$$

Green's thm

e.g. Let S be the circular cone $z=\sqrt{x^2+y^2}$, $0\le z\le 2$, and

$$\vec{F}=(x^2-y)\hat{i}+4z\hat{j}+x^2\hat{k}.$$

Find the circulation of $\vec{F}$ around the boundary of S, C, whose orientation is anticlockwise as viewed from above.

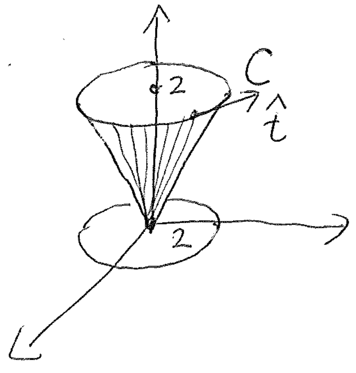
C is described by

$$2=\sqrt{x^2+y^2}, \text{ ie,}$$

its projection onto xy-plane $x^2+y^2=4$.

A standard parametrization is

$$\theta \mapsto (2\cos\theta, 2\sin\theta),\quad \theta\in[0,2\pi].$$



So, C is parametrized by

$$\vec{r}(\theta): \theta\mapsto(2\cos\theta, 2\sin\theta, 2)$$

$$\vec{r}\,'(\theta)=-2\sin\theta\,\hat{i}+2\cos\theta\,\hat{j}$$ points to the anticlockwise direction as viewed from above.

$$\hat{t}=\frac{-2\sin\theta\,\hat{i}+2\cos\theta\,\hat{j}}{\sqrt{(-2\sin\theta)^2+(2\cos\theta)^2}}=-\sin\theta\,\hat{i}+\cos\theta\,\hat{j}$$

On the other hand, S is described by

$$(x,y)\in D_2 \mapsto (x,y,\sqrt{x^2+y^2})$$

$$\vec{r}_x\times\vec{r}_y=\frac{-x}{\sqrt{x^2+y^2}}\hat{i}+\frac{-y}{\sqrt{x^2+y^2}}\hat{j}+\hat{k}$$

the last component is $\hat{k}$, so $\vec{r}_x\times\vec{r}_y$ points upward, and you can check it is consistent with $\hat{t}$, so

$$\hat{n}=+\frac{\vec{r}_x\times\vec{r}_y}{|\vec{r}_x\times\vec{r}_y|}.$$

$$\nabla\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ x^2-y & 4z & x^2\end{vmatrix}=-4\hat{i}-2x\hat{j}+\hat{k}.$$

Things are ready. We'll present 3 way to evaluate

$$\oint_C \vec{F}\cdot d\vec{r}$$

(1st) Use definition.

$$\vec{F}(\vec{r}(\theta)) = (4\cos^2\theta - 2\sin\theta)\,\hat{i} + 4\times 2\,\hat{j} + 4\cos^2\theta\,\hat{k}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left((4\cos^2\theta - 2\sin\theta)\,\hat{i} + 8\,\hat{j} + 4\cos^2\theta\,\hat{k}\right)\cdot (-2\sin\theta\,\hat{i} + 2\cos\theta\,\hat{j} + 0\,\hat{k})\,d\theta$$

$$= \int_0^{2\pi} (-8\cos^2\theta\sin\theta + 4\sin^2\theta)\,d\theta$$

$$= 0 + 2\pi\times 4\times \frac{1}{2} = 4\pi.$$

(2nd) Use Stokes' thm taking S the circular cone.

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot\hat{n}\,d\sigma$$

$$= \iint_{D_2} (-4\hat{i} - 2x\hat{j} + \hat{k})\cdot \vec{r}_x\times\vec{r}_y\,dA(x,y)$$

$$= \iint_{D_2} (-4\hat{i} - 2x\hat{j} + \hat{k})\cdot\left(\frac{-x}{\sqrt{x^2+y^2}}\hat{i} + \frac{-y}{\sqrt{x^2+y^2}}\hat{j} + 1\right)dA(x,y)$$

$$= \iint_{D_2} \left(\frac{4x}{\sqrt{x^2+y^2}} + \frac{2xy}{\sqrt{x^2+y^2}} + 1\right)dA(x,y)$$

$$= \int_0^{2\pi}\int_0^2 (4\cos\theta + 2r\cos\theta\sin\theta + 1)\,r\,dr\,d\theta$$

$$= 4\pi.$$

(3rd) Use Stokes' thm taking the disk K to be the surface.
Observe that C also bounds the disk

$$K = \{(x,y,2) : (x,y)\in D_2\}$$

the normal on K should be $(0,0,1) = \hat{k}$ so that it is consistent with the orientation of C.

$\vec{r}_x \times \vec{r}_y = \hat{k}$ here, so

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_K \nabla \times \vec{F} \cdot \hat{n} \, d\sigma$$

$$= \iint_{D_2} (-4\hat{i} - 2x\hat{j} + \hat{k}) \cdot \hat{k} \, dA(x,y)$$

$$= \iint_{D_2} 1 \, dA(x,y)$$

$$= \pi 2^2 \quad (\text{area of } D_2)$$

$$= 4\pi \quad \#$$